

Improvement upon Mahler's transference theorem. *

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Abstract

In this paper we obtain new transference theorems improving some classical theorems which belong to Kurt Mahler. We formulate those theorems in terms of consecutive minima of pseudo-compound parallelepipeds.

1 Introduction

This paper is devoted to an improvement upon Mahler's theorem published in 1939 in [1, 2], which implies many classical transference theorems. For instance, it implies Khintchine's transference principle [3] connecting the problem of simultaneous approximation to real numbers $\theta_1, \dots, \theta_n$ with the problem of approximating zero with the values of the linear form $\theta_1 x_1 + \dots + \theta_n x_n + x_{n+1}$ at integer points.

Khintchine's transference principle connects the existence of an integer solution to the system of inequalities

$$0 < |x_{n+1}| \leq X, \quad \max_{1 \leq i \leq n} |x_{n+1} \theta_i - x_i| \leq Y \quad (1)$$

with the existence of an integer solution to the system of inequalities

$$0 < \max_{1 \leq i \leq n} |x_i| \leq U, \quad |\theta_1 x_1 + \dots + \theta_n x_n + x_{n+1}| \leq V, \quad (2)$$

where X, Y, U, V are positive real numbers. These two problems are dual in the following sense. Set

$$\begin{aligned} f_i(x_1, \dots, x_{n+1}) &= x_i - \theta_i x_{n+1}, \quad i = 1, \dots, n, \\ f_{n+1}(x_1, \dots, x_{n+1}) &= x_{n+1} \end{aligned} \quad (3)$$

and

$$\begin{aligned} g_i(x_1, \dots, x_{n+1}) &= x_i, \quad i = 1, \dots, n, \\ g_{n+1}(x_1, \dots, x_{n+1}) &= \theta_1 x_1 + \dots + \theta_n x_n + x_{n+1}. \end{aligned} \quad (4)$$

It is easy to see that f_1, \dots, f_{n+1} and g_1, \dots, g_{n+1} are dual bases of the space of linear forms in \mathbb{R}^{n+1} , i.e. the matrices of their coefficients F and G (the coefficients of the i -th form are

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written in the i -th row) satisfies the relation $FG^\top = I$, where I is the identity matrix and G^\top denotes the transpose of G . This means that the above two problems are dual.

Note that the relation $FG^\top = I$ is equivalent to $F^\top G = I$, and also to the fact that the bilinear form

$$\Phi(u_1, \dots, u_{n+1}, v_1, \dots, v_{n+1}) = \sum_{i=1}^{n+1} f_i(u_1, \dots, u_{n+1}) g_i(v_1, \dots, v_{n+1})$$

can be written as

$$\Phi(u_1, \dots, u_{n+1}, v_1, \dots, v_{n+1}) = \sum_{i=1}^{n+1} u_i v_i. \quad (5)$$

This point of view led Mahler to the following ‘theorem on a bilinear form’ which has become classical.

Theorem A (K. Mahler, 1937). *Consider two d -tuples of linear forms in d variables:*

$f_1(\mathbf{u}), \dots, f_d(\mathbf{u})$ in $\mathbf{u} \in \mathbb{R}^d$ with matrix F , $\det F \neq 0$, and

$g_1(\mathbf{v}), \dots, g_d(\mathbf{v})$ in $\mathbf{v} \in \mathbb{R}^d$ with matrix G , $\det G = D \neq 0$.

Suppose that the bilinear form

$$\Phi(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^d f_i(\mathbf{u}) g_i(\mathbf{v}) \quad (6)$$

has integer coefficients. Suppose also that the system of inequalities

$$|f_i(\mathbf{u})| \leq \lambda_i, \quad i = 1, \dots, d \quad (7)$$

admits a nonzero solution in \mathbb{Z}^d . Then so does the system of inequalities

$$|g_i(\mathbf{v})| \leq (d-1)\lambda/\lambda_i, \quad i = 1, \dots, d, \quad (8)$$

where

$$\lambda = \left(|D| \prod_{i=1}^d \lambda_i \right)^{\frac{1}{d-1}}. \quad (9)$$

Theorem A was improved by the first author in [4, 5, 6] for particular cases corresponding to the problems concerning different types of Diophantine exponents. In this paper we improve Theorem A for the arbitrary case. Moreover, we also describe a family of systems analogous to (8), s.t. each system in this family admits a nonzero integer solution, provided so does the system (7). Besides that, we prove the existence of several distinct solutions to (8), among which there are $d-1$ linearly independent ones. The most convenient way to formulate these results is to use consecutive minima of pseudo-compound parallelepipeds.

2 Transference principle and consecutive minima of pseudo-compound parallelepipeds

We remind the definitions of consecutive minima and of pseudo-compound parallelepipeds (see also [7]).

Definition 1. Let M be a convex body in \mathbb{R}^d , symmetric w.r.t. the origin. Let Λ be a d -dimensional lattice in \mathbb{R}^d . Then the k -th successive minimum $\mu_k(M, \Lambda)$ of M w.r.t. Λ is defined as the minimal positive μ such that μM contains k linearly independent points of Λ .

Definition 2. Let h_1, \dots, h_d be d linear forms in \mathbb{R}^d with matrix H , $\det H = 1$, and let h_1^*, \dots, h_d^* be the dual set of linear forms, i.e. $\langle h_i, h_j^* \rangle = \delta_{ij}$, where $\langle \cdot, \cdot \rangle$ denotes inner product. Given positive numbers η_1, \dots, η_d , consider the parallelepiped

$$\Pi = \left\{ \mathbf{z} \in \mathbb{R}^d \mid |h_i(\mathbf{z})| \leq \eta_i, \ i = 1, \dots, d \right\}.$$

Then the parallelepiped

$$\Pi^* = \left\{ \mathbf{z} \in \mathbb{R}^d \mid |h_i^*(\mathbf{z})| \leq \frac{1}{\eta_i} \prod_{j=1}^d \eta_j, \ i = 1, \dots, d \right\}$$

is called *pseudo-compound* for Π .

Let us reformulate Theorem A in terms of pseudo-compound parallelepipeds and their consecutive minima. We shall do it in two steps.

First, let us show that D can be considered to be equal to 1. For each $i = 1, \dots, d$ set

$$f'_i = D^{1/d} f_i, \quad g'_i = D^{-1/d} g_i, \quad \lambda'_i = D^{1/d} \lambda_i, \quad \lambda' = \lambda.$$

It can be easily verified that substitution of $f_i, g_i, \lambda_i, \lambda, D$ with $f'_i, g'_i, \lambda'_i, \lambda', 1$ respectively preserves the statement of Theorem A. Hence, indeed, we can set $D = 1$. Which will be assumed throughout the rest of the paper.

Let us now consider the lattices $F\mathbb{Z}^d$ and $G\mathbb{Z}^d$. The relation (6) means that each of these lattices is a sublattice of the other's dual. We remind the definition.

Definition 3. Let Λ be a d -dimensional lattice in \mathbb{R}^d . Let $\langle \cdot, \cdot \rangle$ denote inner product in \mathbb{R}^d . Then the lattice

$$\Lambda^* = \left\{ \mathbf{z} \in \mathbb{R}^d \mid \langle \mathbf{z}, \mathbf{w} \rangle \in \mathbb{Z} \text{ for all } \mathbf{w} \in \Lambda \right\}$$

is called *dual* for Λ .

Set $\Lambda = G\mathbb{Z}^d$ and consider the parallelepiped

$$\Pi = \left\{ \mathbf{z} = (z_1 \dots z_d)^\top \in \mathbb{R}^d \mid |z_i| \leq \lambda / \lambda_i, \ i = 1, \dots, d \right\},$$

where λ is defined by (9) with $D = 1$, i.e. $\lambda = \left(\prod_{i=1}^d \lambda_i \right)^{\frac{1}{d-1}}$.

Then $\det \Lambda = 1$, $F\mathbb{Z}^d \subseteq \Lambda^*$ and

$$\Pi^* = \left\{ \mathbf{z} = (z_1 \dots z_d)^\top \in \mathbb{R}^d \mid |z_i| \leq \lambda_i, \ i = 1, \dots, d \right\}.$$

Thus, Theorem A actually claims the existence of a nonzero point of Λ in $(d-1)\Pi$ provided there is a nonzero point of a sublattice of Λ^* in Π^* . Clearly, in this statement the words “of a sublattice” can be omitted. Besides that, the presence of a nonzero lattice point inside a parallelepiped means exactly that its first minimum w.r.t. this lattice does not exceed 1. We get the following reformulation of Theorem A.

Theorem B. *Suppose Λ is a d -dimensional lattice in \mathbb{R}^d with covolume 1 and let Π be an $\mathbf{0}$ -symmetric parallelepiped with facets parallel to coordinate hyperplanes. Then*

$$\mu_1(\Pi^*, \Lambda^*) \leq 1 \implies \mu_1(\Pi, \Lambda) \leq d - 1.$$

Note that for each operator $A \in \text{SL}_d(\mathbb{R})$ we have

$$(A\Pi)^* = (A^*)^{-1}\Pi^* \quad \text{and} \quad (A\Lambda)^* = (A^*)^{-1}\Lambda^*,$$

where A^* is the conjugate for A . Therefore, we can map Λ onto \mathbb{Z}^d and thus get another reformulation of Theorem A, “dual” to the formulation of Theorem B, but slightly more concise.

Theorem C. *Let Π be an arbitrary $\mathbf{0}$ -symmetric parallelepiped in \mathbb{R}^d . Then*

$$\mu_1(\Pi^*, \mathbb{Z}^d) \leq 1 \implies \mu_1(\Pi, \mathbb{Z}^d) \leq d - 1.$$

At the same time Mahler [2] proved a theorem concerning *all* of the consecutive minima, which can be formulated as follows.

Theorem D (K. Mahler, 1938). *Let Π be an arbitrary $\mathbf{0}$ -symmetric parallelepiped in \mathbb{R}^d . Then*

$$\frac{2^d}{d \text{vol } \Pi} \leq \mu_k(\Pi^*, \mathbb{Z}^d) \mu_{d+1-k}(\Pi, \mathbb{Z}^d) \leq \frac{2^d d!}{\text{vol } \Pi}. \quad (10)$$

Combining this statement for $k = 1$ with Minkowski’s theorem on consecutive minima, which claims that

$$\frac{2^d}{d! \text{vol } \Pi} \leq \prod_{i=1}^d \mu_i(\Pi, \mathbb{Z}^d) \leq \frac{2^d}{\text{vol } \Pi}, \quad (11)$$

we get the following improvement of Theorem C.

Theorem E. *Let Π be an arbitrary $\mathbf{0}$ -symmetric parallelepiped in \mathbb{R}^d . Let*

$$\mu_1(\Pi^*, \mathbb{Z}^d) \leq 1 \quad \text{and} \quad \mu_1(\Pi, \mathbb{Z}^d) \geq 1.$$

Then

$$\mu_k(\Pi, \mathbb{Z}^d) \leq d^{\frac{1}{d-k}}, \quad k = 1, \dots, d - 1.$$

One of the main results of this paper is the following improvement of Theorem E.

Theorem 1. *Let Π be an arbitrary $\mathbf{0}$ -symmetric parallelepiped in \mathbb{R}^d . Let*

$$\mu_1(\Pi^*, \mathbb{Z}^d) \leq 1 \quad \text{and} \quad \mu_1(\Pi, \mathbb{Z}^d) \geq 1.$$

Then

$$\mu_k(\Pi, \mathbb{Z}^d) \leq d^{\frac{1}{2(d-k)}}, \quad k = 1, \dots, d - 1. \quad (12)$$

For $k = 2$ we prove a stronger inequality.

Theorem 2. *Let Π be an arbitrary $\mathbf{0}$ -symmetric parallelepiped in \mathbb{R}^d . Let*

$$\mu_1(\Pi^*, \mathbb{Z}^d) \leq 1 \quad \text{and} \quad \mu_1(\Pi, \mathbb{Z}^d) > 1.$$

Then

$$\mu_2(\Pi, \mathbb{Z}^d) \leq c_d, \tag{13}$$

where c_d is the positive root of the polynomial $t^{2(d-1)} - (d-1)t^2 - 1$.

It can be easily shown that

$$d^{\frac{1}{2(d-1)}} < c_d < d^{\frac{1}{2(d-2)}}. \tag{14}$$

Thus, indeed, inequality (13) is stronger than (12) for $k = 2$. Besides that, it follows from (14) that

$$c_d = 1 + \frac{\ln d}{2d} + O\left(\frac{\ln^2 d}{d^2}\right) \quad \text{as} \quad d \rightarrow \infty.$$

For $d = 3$ we prove inequalities which are stronger than (12) and (13), and which are moreover precise.

Theorem 3. *Let Π be an arbitrary $\mathbf{0}$ -symmetric parallelepiped in \mathbb{R}^3 . Let*

$$\mu_1(\Pi^*, \mathbb{Z}^3) \leq 1 \quad \text{and} \quad \mu_1(\Pi, \mathbb{Z}^3) > 1.$$

Then

$$\mu_1(\Pi, \mathbb{Z}^3) \leq 2/\sqrt{3} \quad \text{and} \quad \mu_2(\Pi, \mathbb{Z}^3) \leq 5/4.$$

Moreover, the constants $2/\sqrt{3}$ and $5/4$ are exact.

Remark 1. Theorems 1, 2, 3 can be formulated in the likeness of Theorem B. Then we should substitute $\mu_1(\Pi, \mathbb{Z}^d)$ with $\mu_1(\Pi, \Lambda)$, and $\mu_1(\Pi^*, \mathbb{Z}^d)$ with $\mu_1(\Pi^*, \Lambda^*)$.

Theorems 2 and 3 will be obtained as a consequence of an observation which is actually a family of transference theorems.

3 A family of transference theorems

Roughly speaking, regular transference theorems claim the existence of a lattice point in a set provided there is a lattice point in some other set. We are going to construct a whole *family* of parallelepipeds such that each of them will contain a lattice point.

Let Π be an arbitrary $\mathbf{0}$ -symmetric parallelepiped in \mathbb{R}^d . Then there is an operator $A_\Pi \in \text{GL}_d(\mathbb{R})$ such that $A_\Pi \Pi = [-1, 1]^d$. For each d -tuple $\boldsymbol{\tau} = (\tau_1, \dots, \tau_d) \in \mathbb{R}_{>0}^d$ we set

$$H_{\boldsymbol{\tau}, \Pi} = A_\Pi^{-1} \begin{pmatrix} \tau_1 & 0 & \cdots & 0 \\ 0 & \tau_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tau_d \end{pmatrix} A_\Pi.$$

That is $H_{\boldsymbol{\tau}, \Pi}$ is a composition of a hyperbolic shift and a homothety, and the axes of this hyperbolic shift coincide with those of Π . When clear from the context which parallelepiped is under consideration, we shall write $H_{\boldsymbol{\tau}}$ instead of $H_{\boldsymbol{\tau}, \Pi}$.

Theorem 4. Let Π be an arbitrary $\mathbf{0}$ -symmetric parallelepiped in \mathbb{R}^d . Then for each d -tuple $\boldsymbol{\tau} = (\tau_1, \dots, \tau_d)$ such that

$$\sum_{i=1}^d \tau_i^2 = \prod_{i=1}^d \tau_i^2 \quad (15)$$

we have

$$\mu_1(\Pi^*, \mathbb{Z}^d) \leq 1 \implies \mu_1(H_{\boldsymbol{\tau}}\Pi, \mathbb{Z}^d) \leq 1.$$

Remark 2. For each tuple $(\tau_1, \dots, \tau_d) \in \mathbb{R}_{>0}^d$ there is a unique $\lambda > 0$ such that the tuple $(\lambda\tau_1, \dots, \lambda\tau_d)$ satisfies relation (15).

We now show how to derive Theorem 2 from Theorem 4.

Suppose $\mu_1(\Pi^*, \mathbb{Z}^d) \leq 1$ and $\mu_1(\Pi, \mathbb{Z}^d) > 1$. Then by Theorem 4 for each $\boldsymbol{\tau}$ satisfying (15) the parallelepiped $H_{\boldsymbol{\tau}}\Pi$ contains a nonzero point of \mathbb{Z}^d . Consider the minimal t_1 such that for $\boldsymbol{\tau}_1 = (t_1, \dots, t_1)$ the parallelepiped $H_{\boldsymbol{\tau}_1}\Pi$ contains a nonzero point of \mathbb{Z}^d . Then

$$t_1 = \mu_1(\Pi, \mathbb{Z}^d) > 1.$$

There are no nonzero integer points in the interior of $H_{\boldsymbol{\tau}_1}\Pi$, but there is such a point on its boundary. Let us denote it by \mathbf{v} . Without loss of generality we may suppose \mathbf{v} belongs to the facet intersecting the “first” axis of Π , i.e. the one which is mapped onto the first coordinate axis under the action of A_{Π} (under this action Π turns into $[-1, 1]^d$). Consider the minimal $t_2 \geq t_1$ such that for $\boldsymbol{\tau}_2 = (t_1, t_2, \dots, t_2)$ the parallelepiped $H_{\boldsymbol{\tau}_2}\Pi$ contains a nonzero point of \mathbb{Z}^d different from $\pm\mathbf{v}$. This new point is linearly independent with \mathbf{v} , whence

$$\mu_2(\Pi, \mathbb{Z}^d) \leq t_2.$$

If t_2 is strictly larger than the positive root of the equation

$$t_1^2 t^{2(d-1)} = t_1^2 + (d-1)t^2, \quad (16)$$

then by Remark 2 the interior of $H_{\boldsymbol{\tau}_2}\Pi$ contains a parallelepiped $H_{\boldsymbol{\tau}}\Pi$ (homothetic to $H_{\boldsymbol{\tau}_2}\Pi$) with $\boldsymbol{\tau}$ satisfying (15). But this $H_{\boldsymbol{\tau}}\Pi$ does not contain any nonzero integer point, since there are no such points in the interior of $H_{\boldsymbol{\tau}_2}\Pi$. This contradicts Theorem 4 and, therefore, t_2 does not exceed the positive root of (16). Observe that this root decreases as t_1 grows, and by our assumption $t_1 > 1$. Hence t_2 , as well as $\mu_2(\Pi, \mathbb{Z}^d)$, does not exceed the positive root of the polynomial $t^{2(d-1)} - (d-1)t^2 - 1$.

Thus, Theorem 2 indeed follows from Theorem 4. Theorem 4 itself will be proved in Section 7.

4 Main tool: section-dual bodies

Here we describe the main construction which allows proving Theorems 4 and 3.

Let Π be an arbitrary $\mathbf{0}$ -symmetric parallelepiped in \mathbb{R}^d . For each $\mathbf{e} \in \mathbb{R}^d$ we shall use $\text{vol}_{\mathbf{e}}(\Pi)$ to denote the $(d-1)$ -dimensional volume of the intersection of Π with the orthogonal complement to $\mathbb{R}\mathbf{e}$. We shall also use \mathcal{S}^{d-1} to denote the (Euclidean) unit sphere in \mathbb{R}^d .

Definition 4. The set

$$\Pi^\wedge = \{ \lambda \mathbf{e} \mid \mathbf{e} \in \mathcal{S}^{d-1}, 0 \leq \lambda \leq 2^{1-d} \text{vol}_{\mathbf{e}}(\Pi) \}$$

is called *section-dual* for Π .

As a separate concept section-dual bodies were apparently considered first by Lutwak [8]. However in his definition there is no factor like 2^{1-d} and he used the term “intersection body”. For us the factor 2^{1-d} is apt from the point of view of Minkowski’s two theorems: convex body theorem we use to prove statement 1 of Lemma 1, and theorem on consecutive minima we use to prove Lemma 2 (see below).

The following statement is a particular case of the classical Busemann theorem (see [9]).

Proposition 1. Π^\wedge is convex and $\mathbf{0}$ -symmetric.

In [5] the following properties of section-dual sets are proved.

Lemma 1 (see [5]).

1. $\mu_1(\Pi^\wedge, \mathbb{Z}^d) \leq 1 \implies \mu_1(\Pi, \mathbb{Z}^d) \leq 1$.
2. Let $A \in \text{GL}_d(\mathbb{R})$. Then $(A\Pi)^\wedge = A'(\Pi^\wedge)$, where A' is the cofactor matrix of A , i.e. $A' = (\det A)(A^*)^{-1}$.

Statement 1 of Lemma 1 gives us a hint about how to prove Theorem 4. It suffices to show that for each $\mathbf{0}$ -symmetric parallelepiped Π and each $\boldsymbol{\tau} = (\tau_1, \dots, \tau_d)$ satisfying (15) we have

$$\Pi^* \subset (H_{\boldsymbol{\tau}}\Pi)^\wedge. \quad (17)$$

However, to prove Theorem 1 we shall need an enhanced version of statement 1 of Lemma 1.

Lemma 2. Let Π be an arbitrary $\mathbf{0}$ -symmetric parallelepiped in \mathbb{R}^d . Then

$$\mu_1(\Pi^\wedge, \mathbb{Z}^d) \leq 1 \implies \prod_{k=1}^{d-1} \mu_k(\Pi, \mathbb{Z}^d) \leq 1.$$

Proof. Suppose $\mu_1(\Pi^\wedge, \mathbb{Z}^d) \leq 1$. Then there is a (nonzero) primitive integer point \mathbf{v} in Π^\wedge . By the definition of section-dual set this means that

$$\text{vol}_{\mathbf{v}}(\Pi) \geq 2^{d-1}|\mathbf{v}|.$$

Consider the $(d-1)$ -dimensional subspace $\mathcal{L}_{\mathbf{v}}$ orthogonal to \mathbf{v} and set

$$S_{\mathbf{v}} = \Pi \cap \mathcal{L}_{\mathbf{v}}, \quad \Lambda_{\mathbf{v}} = \mathbb{Z}^d \cap \mathcal{L}_{\mathbf{v}}.$$

Then, up to sign, \mathbf{v} coincides with the cross product of any $d-1$ vectors which make a basis of $\Lambda_{\mathbf{v}}$. Hence

$$\det \Lambda_{\mathbf{v}} = |\mathbf{v}| \leq 2^{1-d} \text{vol}_{\mathbf{v}}(\Pi) = 2^{1-d} \text{vol}(S_{\mathbf{v}}).$$

Applying Minkowski’s theorem on consecutive minima we get

$$\prod_{k=1}^{d-1} \mu_k(\Pi, \mathbb{Z}^d) \leq \prod_{k=1}^{d-1} \mu_k(S_{\mathbf{v}}, \Lambda_{\mathbf{v}}) \leq \frac{2^{d-1} \det \Lambda_{\mathbf{v}}}{\text{vol}(S_{\mathbf{v}})} \leq 1.$$

□

5 Section-dual for unit cube

Set $\mathcal{B}_d = [-1, 1]^d$. In other words \mathcal{B}_d is the unit ball in sup-norm. Due to Vaaler's theorem (see [10]) the volume of any $(d-1)$ -dimensional central section of \mathcal{B}_d is not less than 2^{d-1} . Hence \mathcal{B}_d^\wedge contains a Euclidean ball of radius 1, and we get the following statement.

Lemma 3. $\mathcal{B}_d^* = \mathcal{B}_d \subset \sqrt{d} \mathcal{B}_d^\wedge$.

Corollary 1. *For each 0-symmetric parallelepiped Π we have*

$$\Pi^* \subset \sqrt{d} \Pi^\wedge.$$

Proof. Consider $A \in \text{GL}_d(\mathbb{R})$ such that $\Pi = A\mathcal{B}_d$. Then by Lemma 3 and statement 2 of Lemma 1

$$\Pi^* = (A\mathcal{B}_d)^* = A'\mathcal{B}_d^* \subset A'(\sqrt{d}\mathcal{B}_d^\wedge) = \sqrt{d}(A\mathcal{B}_d)^\wedge = \sqrt{d}\Pi^\wedge.$$

□

In order to prove Theorems 4, 3, let us reformulate (17) in terms of the properties of \mathcal{B}_d^\wedge .

Lemma 4. *For each 0-symmetric parallelepiped Π and each d -tuple $\boldsymbol{\tau} = (\tau_1, \dots, \tau_d)$ the inclusion (17) is equivalent to*

$$\left(\prod_{i=1}^d \tau_i\right)^{-1} \begin{pmatrix} \tau_1 \\ \vdots \\ \tau_d \end{pmatrix} \in \mathcal{B}_d^\wedge. \quad (18)$$

Proof. Consider the same $A = A_\Pi$ as in Section 3, so that $A\Pi = \mathcal{B}_d$. Then

$$A'(\Pi^*) = (A\Pi)^* = \mathcal{B}_d^* = \mathcal{B}_d$$

and by statement 2 of Lemma 1

$$A'((H_{\boldsymbol{\tau}}\Pi)^\wedge) = (AH_{\boldsymbol{\tau}}A^{-1}A\Pi)^\wedge = (D_{\boldsymbol{\tau}}\mathcal{B}_d)^\wedge = D_{\boldsymbol{\tau}}'\mathcal{B}_d^\wedge,$$

where

$$D_{\boldsymbol{\tau}} = AH_{\boldsymbol{\tau}}A^{-1} = H_{\boldsymbol{\tau}, \mathcal{B}_d} = \begin{pmatrix} \tau_1 & 0 & \cdots & 0 \\ 0 & \tau_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tau_d \end{pmatrix}.$$

Hence (17) is equivalent to

$$\frac{1}{\det D_{\boldsymbol{\tau}}} D_{\boldsymbol{\tau}} \mathcal{B}_d \subset \mathcal{B}_d^\wedge.$$

And this inclusion in virtue of convexity and symmetry w.r.t. coordinate hyperplanes of both \mathcal{B}_d and \mathcal{B}_d^\wedge is equivalent to the fact that the vertex of

$$\frac{1}{\det D_{\boldsymbol{\tau}}} D_{\boldsymbol{\tau}} \mathcal{B}_d$$

with positive coordinates lies in \mathcal{B}_d^\wedge . But this is exactly what (18) states. □

Corollary 2. *If $\mu_1(\Pi^*, \mathbb{Z}^d) \leq 1$, then $\mu_1(H_{\boldsymbol{\tau}}\Pi, \mathbb{Z}^d) \leq 1$ for every $\boldsymbol{\tau} = (\tau_1, \dots, \tau_d)$ satisfying (18).*

Proof. If (18) holds, then by Lemma 4 we also have (17). Taking into account statement 1 of Lemma 1 we get the following chain of implications

$$\mu_1(\Pi^*, \mathbb{Z}^d) \leq 1 \implies \mu_1((H_{\boldsymbol{\tau}}\Pi)^\wedge, \mathbb{Z}^d) \leq 1 \implies \mu_1(H_{\boldsymbol{\tau}}\Pi, \mathbb{Z}^d) \leq 1.$$

□

6 Proof of Theorem 1

Having Lemma 2 and Corollary 1, it is quite easy to prove Theorem 1. Indeed, those statements immediately imply the implications

$$\begin{aligned} \mu_1(\Pi^*, \mathbb{Z}^d) \leq 1 &\implies \mu_1(\sqrt{d}\Pi^\wedge, \mathbb{Z}^d) \leq 1 \implies \mu_1\left(\left(d^{\frac{1}{2(d-1)}}\Pi\right)^\wedge, \mathbb{Z}^d\right) \leq 1 \implies \\ &\implies \prod_{k=1}^{d-1} \mu_k\left(d^{\frac{1}{2(d-1)}}\Pi, \mathbb{Z}^d\right) \leq 1 \implies \prod_{k=1}^{d-1} \mu_k(\Pi, \mathbb{Z}^d) \leq \sqrt{d}. \end{aligned} \quad (19)$$

Furthermore,

$$\mu_1(\Pi, \mathbb{Z}^d) \leq \dots \leq \mu_{d-1}(\Pi, \mathbb{Z}^d),$$

so within the assumption $\mu_1(\Pi, \mathbb{Z}^d) \geq 1$ the latter inequality in (19) implies that for each $k = 1, \dots, d-1$ we have

$$\mu_k(\Pi, \mathbb{Z}^d) \leq d^{\frac{1}{2(d-k)}}.$$

This proves Theorem 1.

7 Proof of Theorem 4 and its slightly stronger version

As it was said in the previous Section, Vaaler's theorem implies that \mathcal{B}_d^\wedge contains a Euclidean ball of radius 1. Suppose $\boldsymbol{\tau} = (\tau_1, \dots, \tau_d)$ satisfies (15). Then the Euclidean norm of the point

$$\left(\prod_{i=1}^d \tau_i\right)^{-1} \begin{pmatrix} \tau_1 \\ \vdots \\ \tau_d \end{pmatrix} \quad (20)$$

is equal to 1. Hence $\boldsymbol{\tau}$ satisfies (18). It remains to apply Corollary 2.

Theorem 4 is proved.

Theorem 4 is not sharp: we lose sharpness at least when we approximate \mathcal{B}_d^\wedge with the Euclidean unit ball. However, we can confine ourselves with Corollary 2 and get a stronger statement immediately. Set

$$v_{\boldsymbol{\tau}} = 2^{1-d} \text{vol} \left\{ (z_1 \dots z_d)^\top \in \mathcal{B}_d \mid \sum_{i=1}^d \tau_i z_i = 0 \right\},$$

i.e. $v_{\boldsymbol{\tau}}$ is the normalized (in view of Minkowski's theorems) volume of $(d-1)$ -dimensional central section of \mathcal{B}_d orthogonal to $(\tau_1 \dots \tau_d)^\top$. The immediate application of Corollary 2 gives us the following statement, stronger than Theorem 4.

Theorem 5. *Let Π be an arbitrary $\mathbf{0}$ -symmetric parallelepiped in \mathbb{R}^d . Then for each d -tuple $\boldsymbol{\tau} = (\tau_1, \dots, \tau_d)$ such that*

$$\sum_{i=1}^d \tau_i^2 = v_{\boldsymbol{\tau}}^2 \prod_{i=1}^d \tau_i^2, \quad (21)$$

we have

$$\mu_1(\Pi^*, \mathbb{Z}^d) \leq 1 \implies \mu_1(H_{\boldsymbol{\tau}}\Pi, \mathbb{Z}^d) \leq 1.$$

However, besides Vaaler's theorem there is also Ball's theorem (see [11]), which estimates the volume of any $(d-1)$ -dimensional central section of \mathcal{B}_d from above by $2^{d-1}\sqrt{2}$. Thus, in each dimension and for each $\boldsymbol{\tau}$ we have

$$1 \leq v_{\boldsymbol{\tau}} \leq \sqrt{2},$$

and it can be easily seen that both boundaries are attained. This implies that in each dimension the Banach–Mazur distance between the spaces corresponding to \mathcal{B}_d^\wedge and to the Euclidean unit ball is equal to $\sqrt{2}$. Hence substituting $v_{\boldsymbol{\tau}}$ with 1 does not weaken the statement too much, but it makes it sufficiently simpler, for the dependence of $v_{\boldsymbol{\tau}}$ on $\boldsymbol{\tau}$ for arbitrary d is rather complicated.

As for fixed dimensions, for instance, $d = 3$, we can use Corollary 2 (and thus, Theorem 5) explicitly, without approximating \mathcal{B}_d^\wedge with a unit ball, and obtain sharp inequalities.

8 Three-dimensional case. Proof of Theorem 3

For $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ let us set

$$\mathbf{v}_{\boldsymbol{\tau}} = \frac{1}{\tau_1 \tau_2 \tau_3} \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix}$$

By the definition of section-dual set the relation (18) means exactly that the Euclidean norm $|\mathbf{v}_{\boldsymbol{\tau}}|$ does not exceed the area of the central section of \mathcal{B}_3 orthogonal to $\mathbf{v}_{\boldsymbol{\tau}}$ divided by four.

The next statement is a simple school geometry exercise.

Lemma 5. *Given $0 \leq x \leq 1$, the area of the central section of \mathcal{B}_d orthogonal to $(x \ 1 \ 1)^\top$ is equal to $(4-x)\sqrt{2+x^2}$.*

Lemma 6. *Let $\boldsymbol{\tau}' = (2/\sqrt{3}, 2/\sqrt{3}, 2/\sqrt{3})$, $\boldsymbol{\tau}'' = (1, 5/4, 5/4)$. Then the points $\mathbf{v}_{\boldsymbol{\tau}'}$ and $\mathbf{v}_{\boldsymbol{\tau}''}$ lie on the boundary of \mathcal{B}_3^\wedge .*

Proof. It suffices to calculate the areas of central sections of \mathcal{B}_3 orthogonal to $\mathbf{v}_{\boldsymbol{\tau}'}$ and $\mathbf{v}_{\boldsymbol{\tau}''}$ with the help of Lemma 5 and then see that they are equal to $4|\mathbf{v}_{\boldsymbol{\tau}'}|$ and $4|\mathbf{v}_{\boldsymbol{\tau}''}|$, respectively. \square

Let us prove now Theorem 3. The implication

$$\mu_1(\Pi^*, \mathbb{Z}^3) \leq 1 \implies \mu_1(\Pi, \mathbb{Z}^3) \leq 2/\sqrt{3} \quad (22)$$

is an immediate consequence of Lemma 6 and Corollary 2.

Further argument is similar to the one we used when deriving Theorem 2 from Theorem 4. Suppose that $\mu_1(\Pi^*, \mathbb{Z}^3) \leq 1$, but $\mu_1(\Pi, \mathbb{Z}^3) > 1$. Consider the minimal t_1 such that for $\boldsymbol{\tau}_1 = (t_1, t_1, t_1)$ the parallelepiped $H_{\boldsymbol{\tau}_1}\Pi$ contains a nonzero point of \mathbb{Z}^3 . Then $t_1 = \mu_1(\Pi, \mathbb{Z}^3) > 1$.

Denote by \mathbf{v} any integer point lying on the boundary of $H_{\boldsymbol{\tau}_1}\Pi$ (the interior contains no nonzero integer points). As before, let us suppose that \mathbf{v} is on the facet crossing the “first” axis of Π . Consider the minimal $t_2 \geq t_1$ such that for $\boldsymbol{\tau}_2 = (t_1, t_2, t_2)$ the parallelepiped $H_{\boldsymbol{\tau}_2}\Pi$ contains a nonzero integer point other than $\pm\mathbf{v}$. This point is linearly independent with \mathbf{v} , whence

$$\mu_2(\Pi, \mathbb{Z}^3) \leq t_2.$$

If $t_2 > 5/4$, then the interior of $H_{\boldsymbol{\tau}_2}\Pi$ contains a parallelepiped $H_{\boldsymbol{\tau}''}\Pi$, where $\boldsymbol{\tau}'' = (1, 5/4, 5/4)$. There are no nonzero integer points in $H_{\boldsymbol{\tau}''}\Pi$, since there are no such points in the interior of $H_{\boldsymbol{\tau}_2}\Pi$. But Lemma 6 and Corollary 2 imply that such points should exist in $H_{\boldsymbol{\tau}''}\Pi$. The contradiction obtained proves that $t_2 \leq 5/4$, i.e.

$$\begin{cases} \mu_1(\Pi^*, \mathbb{Z}^3) \leq 1 \\ \mu_1(\Pi, \mathbb{Z}^3) > 1 \end{cases} \implies \mu_2(\Pi, \mathbb{Z}^3) \leq 5/4. \quad (23)$$

It remains to show that the inequalities in (22) and (23) are sharp. Let us construct corresponding examples.

Let ε be an arbitrary positive real number, $\varepsilon \leq 1/2$. Consider the parallelepipeds

$$\Pi = \left\{ \mathbf{z} = (z_1 \ z_2 \ z_3)^T \in \mathbb{R}^d \mid |z_i| \leq \varepsilon, \ i = 1, 2, 3 \right\}$$

and

$$\Pi^* = \left\{ \mathbf{z} = (z_1 \ z_2 \ z_3)^T \in \mathbb{R}^d \mid |z_i| \leq \varepsilon^2, \ i = 1, 2, 3 \right\}.$$

Consider also the lattices $\Lambda_1 = A\mathbb{Z}^3$ and $\Lambda_2 = B\mathbb{Z}^3$, where

$$A = \begin{pmatrix} \frac{\varepsilon}{\sqrt{3}} & \frac{2\varepsilon}{\sqrt{3}} & \frac{1}{3\varepsilon^2} \\ \frac{\varepsilon}{\sqrt{3}} & \frac{-\varepsilon}{\sqrt{3}} & \frac{1}{3\varepsilon^2} \\ \frac{-2\varepsilon}{\sqrt{3}} & \frac{-\varepsilon}{\sqrt{3}} & \frac{1}{3\varepsilon^2} \end{pmatrix}, \quad B = \begin{pmatrix} \frac{\varepsilon}{2} & \frac{5\varepsilon}{4} & \frac{1}{3\varepsilon^2} \\ \frac{\varepsilon}{2} & \frac{-3\varepsilon}{4} & \frac{1}{3\varepsilon^2} \\ -\varepsilon & \frac{-\varepsilon}{2} & \frac{1}{3\varepsilon^2} \end{pmatrix},$$

and the corresponding dual lattices $\Lambda_1^* = (A^*)^{-1}\mathbb{Z}^3$ and $\Lambda_2^* = (B^*)^{-1}\mathbb{Z}^3$, where

$$(A^*)^{-1} = \begin{pmatrix} 0 & \frac{1}{\varepsilon\sqrt{3}} & \varepsilon^2 \\ \frac{1}{\varepsilon\sqrt{3}} & \frac{-1}{\varepsilon\sqrt{3}} & \varepsilon^2 \\ \frac{-1}{\varepsilon\sqrt{3}} & 0 & \varepsilon^2 \end{pmatrix}, \quad (B^*)^{-1} = \begin{pmatrix} \frac{1}{12\varepsilon} & \frac{1}{2\varepsilon} & \varepsilon^2 \\ \frac{7}{12\varepsilon} & \frac{-1}{2\varepsilon} & \varepsilon^2 \\ \frac{-2}{3\varepsilon} & 0 & \varepsilon^2 \end{pmatrix}.$$

Let us denote the columns of A , B , $(A^*)^{-1}$, $(B^*)^{-1}$ by \mathbf{a}_i , \mathbf{b}_i , \mathbf{a}_i^* , \mathbf{b}_i^* , $i = 1, 2, 3$. Then

$$\begin{aligned} \Lambda_1 &= \text{span}_{\mathbb{Z}}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3), & \Lambda_1^* &= \text{span}_{\mathbb{Z}}(\mathbf{a}_1^*, \mathbf{a}_2^*, \mathbf{a}_3^*), \\ \Lambda_2 &= \text{span}_{\mathbb{Z}}(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3), & \Lambda_2^* &= \text{span}_{\mathbb{Z}}(\mathbf{b}_1^*, \mathbf{b}_2^*, \mathbf{b}_3^*). \end{aligned}$$

Lemma 7. *Let $\nu_1 = 2/\sqrt{3}$, $\nu_2 = 5/4$. Then*

$$\begin{aligned}\nu_1\Pi \cap \Lambda_1 &= \{\mathbf{0}, \pm\mathbf{a}_1, \pm\mathbf{a}_2, \pm(\mathbf{a}_1 - \mathbf{a}_2)\}, & \text{int}(\nu_1\Pi) \cap \Lambda_1 &= \text{int}(\Pi) \cap \Lambda_2 = \{\mathbf{0}\}, \\ \nu_2\Pi \cap \Lambda_2 &= \{\mathbf{0}, \pm\mathbf{b}_1, \pm\mathbf{b}_2, \pm(\mathbf{b}_1 - \mathbf{b}_2)\}, & \text{int}(\nu_2\Pi) \cap \Lambda_2 &= \Pi \cap \Lambda_2 = \{\mathbf{0}, \pm\mathbf{b}_1\}.\end{aligned}$$

Besides that,

$$\Pi^* \cap \Lambda_1^* = \Pi^* \cap \Lambda_2^* = \{\mathbf{0}, \pm\mathbf{a}_3^*\}, \quad \text{int}(\Pi^*) \cap \Lambda_1^* = \text{int}(\Pi^*) \cap \Lambda_2^* = \{\mathbf{0}\}.$$

Proof. Suppose $\mathbf{a} = k_1\mathbf{a}_1 + k_2\mathbf{a}_2 + k_3\mathbf{a}_3$ with integer k_1, k_2, k_3 and suppose $\mathbf{a} \in \nu_1\Pi$. Then the sup-norm of \mathbf{a} does not exceed $2\varepsilon/\sqrt{3}$. Which implies the inequalities

$$\begin{cases} -1 \leq \frac{k_1}{2} + k_2 + \frac{k_3}{2\sqrt{3}\varepsilon^3} \leq 1 \\ -1 \leq \frac{k_1}{2} - \frac{k_2}{2} + \frac{k_3}{2\sqrt{3}\varepsilon^3} \leq 1 \\ -1 \leq -k_1 - \frac{k_2}{2} + \frac{k_3}{2\sqrt{3}\varepsilon^3} \leq 1 \end{cases} \quad (24)$$

Hence $|k_3| \leq 2\sqrt{3}\varepsilon^3 < 1$, as $\varepsilon \leq 1/2$. Therefore, $k_3 = 0$. Now it follows from (24) that $|k_1| \leq 4/3$ and $|k_2| \leq 4/3$, i.e. $k_1, k_2 \in \{-1, 0, 1\}$. But $k_1 = k_2 = \pm 1$ does not satisfy the first equation. It remains to verify explicitly that $\pm\mathbf{a}_1, \pm\mathbf{a}_2, \pm(\mathbf{a}_1 - \mathbf{a}_2)$ lie on the boundary of $\nu_1\Pi$.

Using similar argument for $\mathbf{b} = k_1\mathbf{b}_1 + k_2\mathbf{b}_2 + k_3\mathbf{b}_3 \in \nu_2\Pi$ we get $k_3 = 0$, $k_1, k_2 \in \{-1, 0, 1\}$. After which it remains to verify that $\pm(\mathbf{b}_1 + \mathbf{b}_2)$ are not in $\nu_2\Pi$, that $\pm\mathbf{b}_2, \pm(\mathbf{b}_1 - \mathbf{b}_2)$ are on the boundary of $\nu_2\Pi$, and that $\pm\mathbf{b}_1$ are on the boundary of Π .

The statements of our Lemma concerning Π^* are proved with the same method, and the argument is very simple due to the inequality $\varepsilon \leq 1/2$ and to the fact that some of the coordinates of $\mathbf{a}_1^*, \mathbf{a}_2^*, \mathbf{b}_2^*$ are zero. \square

Corollary 3. *We have*

$$\mu_1(\Pi^*, \Lambda_1^*) = \mu_1(\Pi^*, \Lambda_2^*) = \mu_1(\Pi, \Lambda_2) = 1, \quad \mu_1(\Pi, \Lambda_1) = 2/\sqrt{3}, \quad \mu_2(\Pi, \Lambda_2) = 5/4.$$

Actually, in view of Remark 1, Corollary 3 is already what we need. But to be complete let us reformulate it. Set

$$\Pi_1 = A^{-1}\Pi, \quad \Pi_2 = B^{-1}\Pi.$$

Then

$$A^{-1}\Lambda_1 = B^{-1}\Lambda_2 = A^*\Lambda_1^* = B^*\Lambda_2^* = \mathbb{Z}^3.$$

We get the following reformulation of Corollary 3.

Corollary 4. *We have*

$$\mu_1(\Pi_1^*, \mathbb{Z}^3) = \mu_1(\Pi_2^*, \mathbb{Z}^3) = \mu_1(\Pi_2, \mathbb{Z}^3) = 1, \quad \mu_1(\Pi_1, \mathbb{Z}^3) = 2/\sqrt{3}, \quad \mu_2(\Pi_2, \mathbb{Z}^3) = 5/4.$$

Thus, we have constructed examples which confirm sharpness of the inequalities (22) and (23), and have proved Theorem 3.

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